

Topological surfaces as gridded surfaces in geometrical spaces.

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Abstract

In this paper we study topological surfaces as gridded surfaces in the 2-dimensional scaffolding of cubic honeycombs in Euclidean and hyperbolic spaces.

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1 Introduction

The category of cubic complexes and cubic maps is similar to the simplicial category. The only difference consists in considering cubes of different dimensions instead of simplexes. In this context, a *cubulation* of a manifold is a cubical complex which is PL homeomorphic to the manifold (see [4], [7], [12]). In this paper we study the realizations of cubulations of manifolds embedded in skeletons (or scaffoldings) of the canonical cubical honeycombs of an euclidean or hyperbolic space.

In [1] it was shown the following theorem:

Theorem 1.1. *Let $M, N \subset \mathbb{R}^{n+2}$, $N \subset M$, be closed and smooth submanifolds of \mathbb{R}^{n+2} such that $\dim(M) = n + 1$ and $\dim(N) = n$. Suppose that N has a trivial normal bundle in M (i.e., N is a two-sided hypersurface of M). Then there exists an ambient isotopy of \mathbb{R}^{n+2} which takes M into the $(n + 1)$ -skeleton of the canonical cubulation \mathcal{C} of \mathbb{R}^{n+2} and N into the n -skeleton of \mathcal{C} . In particular, N can be deformed by an ambient isotopy into a cubical manifold contained in the canonical scaffolding of \mathbb{R}^{n+2} .*

In particular, the previous theorem establishes that smooth knotted surfaces in \mathbb{R}^4 are isotopic to cubulated 2-knots in the 4-dimensional cubic honeycomb. Orientable smooth closed surfaces are cubulated manifolds.

Definition 1.2. Let S be a topological 2-manifold embedded in either \mathbb{R}^n or \mathbb{H}^n . We say that S is a *geometrically gridded surface* if it is contained in the 2-skeleton of either the canonical cubulation $\{4, 3^{n-2}, 4\}$ of \mathbb{R}^n , the hyperbolic cubic honeycomb $\{4, 3, 5\}$ of the hyperbolic space \mathbb{H}^3 or the hyperbolic cubic honeycomb $\{4, 3, 3, 5\}$ of the hyperbolic space \mathbb{H}^4 . Here *geometrically* means that the surface is gridded by isometric pieces of regular euclidean or hyperbolic squares and it is placed in the corresponding skeleton in the scaffolding \mathcal{S}^2 of a regular cubic euclidean or hyperbolic honeycomb \mathcal{C} . If it is understood the type of squares we simply say that S is a *gridded surface*.

A *gridded surface* S on \mathcal{S}^2 of \mathcal{C} in \mathbb{R}^n is a piecewise linear surface such that each linear piece is a unit square with its vertices in the \mathbb{Z}^n -lattice of \mathbb{R}^n .

However in the case of hyperbolic spaces \mathbb{H}^n the description of the vertices is more complicated since the vertices belong to an orbit of a non-abelian discrete group acting by isometries on hyperbolic space \mathbb{H}^n as we will see in the next sections. We remark that our gridded surfaces are in a natural way length spaces [2].

Hilbert (1901, [9]) proved that there is no regular smooth isometric immersion $X : \mathbb{H}^2 \rightarrow \mathbb{R}^3$. Efimov (1961, [5]) generalized this nonexistence theorem of Hilbert to the case of complete surfaces of nonpositive curvature; more precisely, he showed that there is no \mathcal{C}^2 -isometric immersion of a complete, two dimensional, Riemannian manifold $M \subset \mathbb{R}^3$ whose curvature satisfies $K \leq c < 0$. However, J. Nash (1956, [18]) proved that any \mathcal{C}^k -manifold (M^n, g) can be \mathcal{C}^k -isometrically immersed into \mathbb{R}^n where $q \geq \frac{3}{2}n(n+1)(n+9)$ and $k \geq 3$. This implies that the hyperbolic space can be embedded into a high dimensional Euclidean space; for instance, we can find a \mathcal{C}^3 -isometric embedding of \mathbb{H}^2 into \mathbb{R}^{99} (see [8], [18]).

In this paper we study topological surfaces as geometrically gridded surfaces in the scaffolding of cubic honeycombs in Euclidean and hyperbolic spaces. We prove that orientable surfaces with a finite number of ends and arbitrary genus can be gridded in \mathbb{R}^3 and all the surfaces, orientable or not, of arbitrary genus but with a finite number of ends can be gridded in \mathbb{R}^4 . Moreover we show that there exist surfaces that can not be gridded in \mathbb{R}^n . We prove that all topological surfaces, orientable or not, of any genus and set of ends can be gridded in \mathbb{H}^4 and also that the subset of orientable surfaces can be gridded in both \mathbb{H}^3 and \mathbb{H}^4 .

2 Preliminaries

This section consists of two topics. The first studies the type of “scaffolding” (*i.e.* the 2-skeleton of an Euclidean or hyperbolic honeycomb) in which we can embed a surface in order to make it a gridded surface. For this purpose we start studying the regular cubic honeycombs of dimensions 3 and 4 in the Euclidean and hyperbolic cases.

In the second topic we will revise the theorem of topological classification of non-compact surfaces, in particular non-compact surfaces with Cantor sets of ends of planar and nonplanar type, and also with Cantor sets of non-orientable ends.

2.1 Regular cubic honeycombs

We are interested in geometrically regular cubic honeycombs which are geometric spaces filled with hypercubes which are euclidean or hyperbolic hypercubes. We denote these honeycombs by their Schläfli symbols. For a cube the symbol is $\{4, 3\}$. This means that the faces of the regular cube are squares with Schläfli symbol $\{4\}$ and that there are 3 squares around each vertex. These cubic honeycombs have Schläfli symbols which describe their geometry and start by $\{4, 3, \dots\}$.

2.1.1 Euclidean cubic honeycombs $\{4, 3^{n-2}, 4\}$

The *canonical cubulation* \mathcal{C}^n of \mathbb{R}^n is its decomposition into n -dimensional cubes which are the images of the unit n -cube $I^n = [0, 1]^n$ by translations by vectors with integer coefficients. Then all vertices of \mathcal{C}^n have integers in their coordinates.

Any *cubulation* of \mathbb{R}^n is obtained by applying a conformal transformation to the canonical cubulation. Remember that a *conformal transformation* is of the form $x \mapsto \lambda A(x) + a$, where $\lambda \neq 0$, $a \in \mathbb{R}^n$, $A \in SO(n)$.

Any n -cubulation has the same combinatorial structure as honeycomb. The regular hypercubic honeycomb whose Schläfli symbol is $\{4, 3^{n-2}, 4\}$ is a *cubulation* of \mathbb{R}^n which is its decomposition into a collection \mathcal{C}^n of right-angled n -dimensional hypercubes $\{4, 3^{n-2}\}$ called the *cells* such that any two are either disjoint or meet in one common k -face of some dimension k . This provides \mathbb{R}^n with the structure of a cubic complex whose category is similar to the simplicial category PL.

The combinatorial structure of the regular honeycomb $\{4, 3, 4\}$ is as follows: there are 6 edges, 12 squares and 8 cubes which are incident for each vertex and there are 4 squares and 4 cubes which are incident for each edge.

The combinatorial structure of the regular honeycomb $\{4, 3, 3, 4\}$ is as follows: there are 8 edges, 24 squares, 32 cubes and 16 hypercubes which are incident for each vertex; there are 6 squares, 32 cubes and 16 hypercubes which are incident for each edge and there are 4 cubes and 4 hypercubes which are incident for each square.

Definition 2.1. The k -*skeleton* of the canonical cubulation \mathcal{C}^n of \mathbb{R}^n , de-

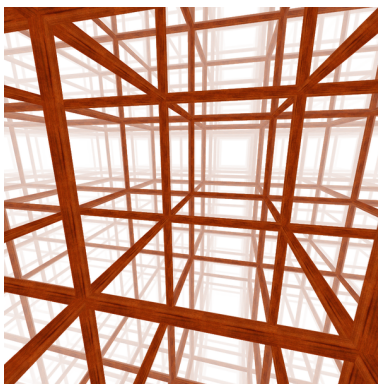


Figure 1: *The 3-dimensional cubic kaleidoscopic honeycomb $\{4, 3, 4\}$. This Figure is courtesy of Roice Nelson [13].*

noted by \mathcal{C}^k , consists of the union of the k -skeletons of the hypercubes in \mathcal{C}^n , *i.e.*, the union of all cubes of dimension k contained in the n -cubes in \mathcal{C}^n . We will call the 2-skeleton \mathcal{C}^2 of \mathcal{C}^n the *canonical scaffolding* of \mathbb{R}^n .

2.1.2 Hyperbolic cubic honeycombs $\{4, 3, 5\}$ and $\{4, 3, 3, 5\}$

A gridded surface is a surface made of congruent squares contained in the corresponding scaffoldings which have disjoint interiors and are glued only in their edges, two squares are either disjoint or share one edge or a vertex. Around each vertex there is a circular circuit of squares (the *squared star* of the vertex). The condition of regularity of platonic solids in the 3-dimensional case implies that the number of squares around each vertex is a constant $k > 2$. If $k = 3$ then the regular gridded surface is the cube $\{4, 3\}$ which is homeomorphic to a sphere \mathbb{S}^2 . If $k = 4$ then the regular surface is the gridded plane $\{4, 4\}$ which is isometric to the Euclidean plane \mathbb{R}^2 . If $k > 4$ then the regular surfaces are the gridded hyperbolic planes which are denoted by $\{4, 5\}, \{4, 6\}, \dots$. These gridded hyperbolic planes are length spaces [2] and are isometric, as length metric spaces to the hyperbolic plane \mathbb{H}^2 .

Analogously, we consider regular gridded 3-manifolds made of congruent cubes in scaffoldings which are disjoint and glued only in their boundary squares, two cubes are either disjoint or meet at a vertex an edge or a square. There is a circular circuit of cubes around each edge and one spher-

ical circuit of cubes around each vertex as PL 3-manifolds. For each edge there is the same number of cubes $k > 2$. If $k = 3$ then the regular gridded 3-manifold is the hypercube $\{4, 3, 3\}$ which is homeomorphic to the 3-sphere \mathbb{S}^3 . If $k = 4$ then the regular gridded 3-manifold is the cubic space $\{4, 3, 4\}$ which is isometric to the Euclidean space \mathbb{R}^3 . If $k = 5$ then the regular gridded 3-manifold is the cubic hyperbolic 3-space $\{4, 3, 5\}$ which is isometric as a length space to the hyperbolic 3-space \mathbb{H}^3 [2].

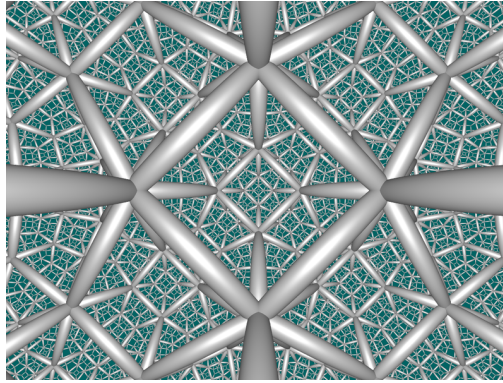


Figure 2: *The 3-dimensional cubic hyperbolic honeycomb $\{4, 3, 5\}$. This Figure is courtesy of Roice Nelson [13].*

Finally we construct regular gridded 4-manifolds made of congruent hypercubes contained in the corresponding scaffoldings which are disjoint and glued only in their boundary cubes, two hypercubes are either disjoint, or meet along a boundary cube of some dimension. There is one circular circuit of hypercubes around each square, one 2-spherical circuit of hypercubes around each edge and one 3-spherical circuit of hypercubes around each edge vertex as PL 4-manifolds.

For each square which is a ridge, there is the same number of hypercubes $k > 3$. If $k = 3$ then the regular gridded 4-manifold is the hypercube $\{4, 3, 3, 3\}$ which is homeomorphic to the 4-sphere \mathbb{S}^4 . If $k = 4$ then the regular 4-manifold is the gridded cubic 4-space $\{4, 3, 3, 4\}$ which is isometric to the Euclidean 4-space \mathbb{R}^4 . If $k = 5$ then the regular 4-manifold is the gridded hyperbolic 5-space $\{4, 3, 3, 5\}$ which is isometric to the hyperbolic 4-space \mathbb{H}^4 .

The combinatorial structure of the regular hyperbolic honeycomb $\{4, 3, 5\}$ (\mathcal{C}) is as follows: there are 12 edges, 30 squares and 20 cubes meeting at every vertex and there are 5 squares and 5 cubes meeting at every edge.

The combinatorial structure of the regular honeycomb $\{4, 3, 3, 5\}$ (\mathcal{C}) is as follows: there are 120 edges, 720 squares, 1200 cubes and 600 hypercubes meeting at every vertex; there are 6 squares, 32 cubes and 16 hypercubes meeting at every edge and there are 5 cubes and 5 hypercubes meeting at every square.

As before, we define the k -skeleton of the hyperbolic honeycomb of \mathbb{H}^n ($n = 3, 4$), denoted by \mathcal{C}^k , consists of the union of the k -skeletons of the hypercubes in \mathcal{C} , *i.e.*, the union of all cubes of dimension k contained in the n -cubes in \mathcal{C} . We will call the 2-skeleton \mathcal{C}^2 of \mathcal{C} the *canonical scaffolding* of \mathbb{H}^n ($n = 3, 4$).

2.2 Taxonomy of topological surfaces

Next we will consider all topological surfaces. First we will remind some definitions.

Let $g(X)$ denote the genus of a compact surface X . The *genus* of the noncompact surface X is by definition

$$g(X) = \max\{g(A) : A \text{ is a compact subsurface of } X\},$$

if this maximum exists, and infinite otherwise ($g(X) = \infty$).

If $g(X) = 0$ we say that the surface X is *planar*. In this case the surface is homeomorphic to an open subset of the plane.

Definition 2.2. There are four orientability classes of noncompact surfaces:

1. If every compact subsurface of a surface X is orientable, then X is *orientable*.
2. If there is no bounded subset A of X such that $X - A$ is orientable, then X is *infinitely nonorientable*.
3. If X does not belong to (1) or (2) and every sufficiently large subsurface of X contains an even number of cross caps, then X is *even non orientable*.

4. If X does not belong to (1) or (2) and every sufficiently large subsurface of X contains an odd number of cross caps, then X is *odd non orientable*.

Definition 2.3. Let X be a surface. An *end* of X is a function e which assigns to each compact subset K of X an unbounded, connected component $e(K)$ of $\text{Cl}(X - K)$ in such a way that if K and L are compact subsets of X and $K \subset L$ then $e(L) \subset e(K)$. $E(X)$ denotes the set of all ends of X .

If X is a noncompact surface, there is a compact subsurface A of X such that each component of $\text{Cl}(X - A)$ is either orientable or infinitely nonorientable, and either planar or of infinite genus. In all that follows A will denote such a subsurface.

Definition 2.4. Let $e \in E(X)$. We say that e is nonorientable (or orientable) if $e(A)$ is infinitely nonorientable (or orientable). And e is planar (or of infinite genus) if $e(A)$ is planar (or of infinite genus).

For any surface X consider the nested triple $(E(X), G(X), O(X))$, where $E(X)$ is the set of ends, $G(X)$ is the subset of $E(X)$ consisting of the ends which are not planar, and $O(X)$ is the subset of $G(X)$ of the non orientable ends.

The next Theorem is by Kerékjártó and Richards, the reader can find it in [14].

Theorem. (Classification theorem for noncompact surfaces). *Let X and Y be two noncompact surfaces of the same genus and orientability class. Then X and Y are homeomorphic if and only if the triads $(E(X), G(X), O(X))$ and $(E(Y), G(Y), O(Y))$ are topologically equivalent.*

Remark 2.5. The condition that X and Y are of the same genus and orientability class guaranties that their bounded parts are homeomorphic. The condition on the subsets of ends makes sure that their asymptotic behavior is the same.

Our goal is to show that any surface is homeomorphic to a gridded surface. In order to prove it, we will use the classification theorem for noncompact surfaces and the theorems of Richards which appear in [14].

Theorem 2.6. *The set of ends of a surface is totally disconnected, separable, and compact. Any compact, separable, totally disconnected space is homeomorphic to a subset of the Cantor set.*

Theorem 2.7. *Let (X, Y, Z) be any triple of compact, separable, totally disconnected spaces with $Z \subset Y \subset X$. Then there is a surface S whose ideal boundary $(E(S), G(S), O(S))$ is topologically equivalent to the triple (X, Y, Z) .*

Theorem 2.8. *Every surface is homeomorphic to a surface formed from a 2-sphere \mathbb{S}^2 by first removing a closed totally disconnected set X from \mathbb{S}^2 , then removing the interiors of a finite or infinite sequence D_1, D_2, \dots of non-overlapping closed discs in $\mathbb{S}^2 - X$, and finally suitably identifying the boundaries of these discs in pairs, (It may be necessary to identify the boundary of one disc with itself to produce an odd "cross cap.") The sequence D_1, D_2, \dots "approaches X " in the sense that, for any open set $U \subset \mathbb{S}^2$ containing X , all but a finite number of the D_i are contained in U .*

Let $X \subset \mathbb{S}^2$ a Cantor set, the orientable noncompact surface $\mathbb{S}^2 - X$ is called *the tree of life*. The *tree of life* can be constructed by an infinite set of pair of pants glued along their boundaries. A pair of pants is by definition the surface with boundary $\mathbb{S}^2 - \{D_1, D_2, D_3\}$ where each D_i is an open disc. Consider the surfaces with boundary Σ_1 and Σ_2 which are the connected sum of a pair of pants with a torus $\Sigma_1 = \mathbb{T}^2 - \{D_1, D_2, D_3\}$ and with a projective plane $\Sigma_2 = \mathbb{P}^2 - \{D_1, D_2, D_3\}$.

A *pruned tree of life* is obtained from the tree of life removing a set of branches (neighborhoods of ends) cut along boundaries of the corresponding pair of pants where these boundaries are identified to points (i.e, we cap some holes of the pair of pants).

For the purpose of this paper we will rephrase the Theorem 2.8, as follows.

Theorem 2.9. *Every surface is homeomorphic to a surface obtained from a pruned tree of life such that a finite or infinite number of pair of pants are interchanged by either the connected sum of a pair of pants with a torus i.e., Σ_1 or the connected sum of pair of pants with a projective plane i.e., Σ_2 .*

For instance, consider the *cylinder*; i.e. the noncompact surface homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Then, it can be obtained from the tree of life if one removes all of its branches except two but and at each boundary component we glue a disk. The gridded surface obtained this way is depicted in Figure 3. In this way, the cylinder is composed by an infinite number of pair of pants.

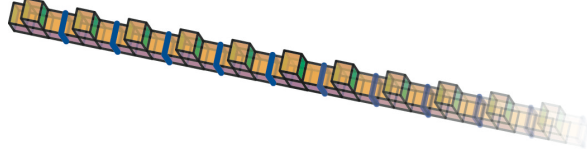


Figure 3: *The gridded cylinder is obtained from the tree of life after pruning and filling the remaining holes with squares (the squares at the top).*

3 Surfaces in $\{4, 3^{n-2}, 4\}$ in \mathbb{R}^n

3.1 Compact surfaces in $\{4, 3, 3, 4\}$ in \mathbb{R}^4

As we mentioned before, we will prove that any connected surface with a finite number of ends is homeomorphic to a gridded surface. We will start considering closed surfaces.

Notice that the 2-sphere \mathbb{S}^2 and the 2-torus \mathbb{T}^2 are homeomorphic to gridded surfaces contained in the scaffolding \mathcal{C}^2 of the canonical cubulation \mathcal{C} of \mathbb{R}^3 in the obvious way. In fact, Consider the unit 3-cube $I^3 = [0, 1]^3$, clearly its boundary ∂I^3 is contained in \mathcal{C}^2 and is homeomorphic to the 2-sphere \mathbb{S}^2 (see Figure 4).

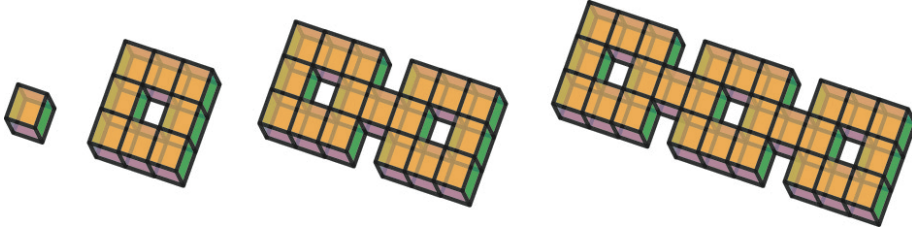


Figure 4: *Closed gridded surfaces in \mathbb{R}^3 .*

The 2-torus \mathbb{T}^2 is the gridded surface $S = \cup_{i=0}^9 F_i$ (see the second image of the Figure 4), where F_i , $i = 0, \dots, 9$ are the following 10 sets which are unions of squares (squared sets):

$$F_0 = \{(x, y, 0) \in \mathbb{R}^3 \mid 0 \leq x, y \leq 3\} \setminus \{(x, y, 0) \in \mathbb{R}^3 \mid 1 < x, y < 2\},$$

$$F_1 = \{(x, y, 1) \in \mathbb{R}^3 \mid 0 \leq x, y \leq 3\} \setminus \{(x, y, 0) \in \mathbb{R}^3 \mid 1 < x, y < 2\},$$

$$\begin{aligned}
F_2 &= \{(0, y, z) \in \mathbb{R}^3 \mid 0 \leq y \leq 3, 0 \leq z \leq 1\}, \\
F_3 &= \{(1, y, z) \in \mathbb{R}^3 \mid 0 \leq y \leq 3, 0 \leq z \leq 1\}, \\
F_4 &= \{(x, 0, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 3, 0 \leq z \leq 1\}, \\
F_5 &= \{(x, 1, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 3, 0 \leq z \leq 1\}, \\
F_6 &= \{(x, 1, z) \in \mathbb{R}^3 \mid 1 \leq x \leq 2, 0 \leq z \leq 1\}, \\
F_7 &= \{(x, 2, z) \in \mathbb{R}^3 \mid 1 \leq x \leq 2, 0 \leq z \leq 1\}, \\
F_8 &= \{(1, y, z) \in \mathbb{R}^3 \mid 1 \leq y \leq 2, 0 \leq z \leq 1\}, \\
F_9 &= \{(2, y, z) \in \mathbb{R}^3 \mid 1 \leq y \leq 2, 0 \leq z \leq 1\}.
\end{aligned}$$

Remark 3.1. Notice that each square F of the canonical cubulation \mathcal{C} of \mathbb{R}^4 (or \mathbb{R}^3) is determined by its barycenter B_F . In fact, consider the unitary canonical vectors on \mathbb{R}^4 : $e_{\pm 1} = (\pm 1, 0, 0, 0)$, $e_{\pm 2} = (0, \pm 1, 0, 0)$, $e_{\pm 3} = (0, 0, \pm 1, 0)$ and $e_{\pm 4} = (0, 0, 0, \pm 1)$. Then

$$F = \{ae_u + be_v : 0 \leq a, b \leq 1\} + w$$

where e_u and e_v ($u, v \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, $|u| \neq |v|$), denote the corresponding unitary canonical vectors and w is a vector with integers in its coordinates, in fact w is a translation vector. Thus $B_F = \frac{1}{2}e_u + \frac{1}{2}e_v + w$.

We identify the squares of the torus with their barycenters, then:

$$\begin{aligned}
F_0 &= (\frac{1}{2}, \frac{1}{2}, 0), (\frac{3}{2}, \frac{1}{2}, 0), (\frac{5}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{3}{2}, 0), (\frac{5}{2}, \frac{3}{2}, 0), (\frac{1}{2}, \frac{5}{2}, 0), (\frac{3}{2}, \frac{5}{2}, 0), (\frac{5}{2}, \frac{5}{2}, 0). \\
F_1 &= (\frac{1}{2}, \frac{1}{2}, 1), (\frac{3}{2}, \frac{1}{2}, 1), (\frac{5}{2}, \frac{1}{2}, 1), (\frac{1}{2}, \frac{3}{2}, 1), (\frac{5}{2}, \frac{3}{2}, 1), (\frac{1}{2}, \frac{5}{2}, 1), (\frac{3}{2}, \frac{5}{2}, 1), (\frac{5}{2}, \frac{5}{2}, 1). \\
F_2 &= (0, \frac{1}{2}, \frac{1}{2}), (0, \frac{3}{2}, \frac{1}{2}), (0, \frac{5}{2}, \frac{1}{2}). \\
F_3 &= (1, \frac{3}{2}, \frac{1}{2}). \\
F_4 &= (2, \frac{3}{2}, \frac{1}{2}). \\
F_5 &= (3, \frac{1}{2}, \frac{1}{2}), (3, \frac{3}{2}, \frac{1}{2}), (3, \frac{5}{2}, \frac{1}{2}). \\
F_6 &= (\frac{1}{2}, 0, \frac{1}{2}), (\frac{3}{2}, 0, \frac{1}{2}), (\frac{5}{2}, 0, \frac{1}{2}). \\
F_7 &= (\frac{3}{2}, 1, \frac{1}{2}). \\
F_8 &= (\frac{3}{2}, 2, \frac{1}{2}). \\
F_9 &= (\frac{1}{2}, 3, \frac{1}{2}), (\frac{3}{2}, 3, \frac{1}{2}), (\frac{5}{2}, 3, \frac{1}{2}).
\end{aligned}$$

The Klein bottle and the real projective plane can not be homeomorphic to gridded surfaces contained in the 2-skeleton of \mathbb{R}^3 (see [17]) but they are homeomorphic to gridded surfaces contained in the 2-skeleton of \mathbb{R}^4 .

Lemma 3.2. *The projective plane (\mathbb{P}^2) is a gridded surface in $\{4, 3, 3, 4\}$ in \mathbb{R}^4 .*

Proof. We construct a gridded version of the crosscap in $\{4, 3, 3, 4\}$ in \mathbb{R}^4 , see Figure 5. In the left we show the projection of the crosscap. In the middle we divide the crosscap in three parts: on the bottom, there is the base which is a cubic box minus two squares, in the middle there is a band and on the top, there is a disk which is the neighborhood of one vertex. In the top right part of the Figure 5, we can find the description of the combinatorial square complex of the crosscap as a disk consisting on 30 squares, such that points in the circle boundary are identified by the antipodal map. On the bottom, there is a Möbius band contained in this crosscap.

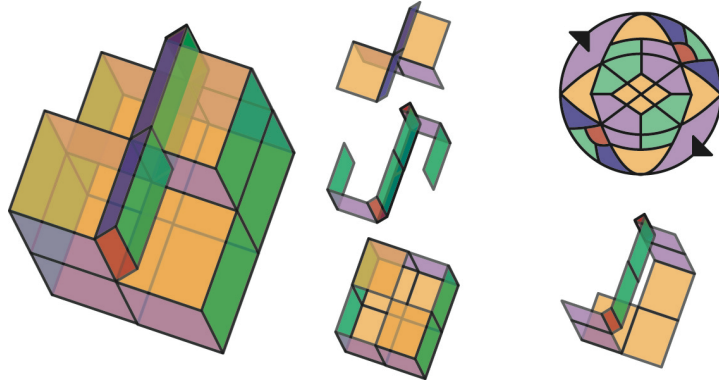


Figure 5: A gridded projective plane in \mathbb{R}^4 . The first figure at the left is the projection of the crosscap into \mathbb{R}^3 .

The gridded crosscap is formed by 30 squares in planes parallel to five of the six coordinate planes in \mathbb{R}^4 and whose barycenters are:

$$XY: \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), \left(\frac{3}{2}, \frac{1}{2}, 0, 0\right), \left(\frac{3}{2}, \frac{3}{2}, 0, 0\right), \left(\frac{1}{2}, \frac{3}{2}, 0, 0\right), \left(\frac{3}{2}, \frac{1}{2}, 1, 0\right), \left(\frac{1}{2}, \frac{3}{2}, 1, 0\right), \\ \left(\frac{1}{2}, \frac{1}{2}, 2, 0\right), \left(\frac{3}{2}, \frac{3}{2}, 2, 0\right).$$

$$XZ: \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{3}{2}, 0\right), \left(\frac{3}{2}, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 1, \frac{3}{2}, 0\right), \left(\frac{3}{2}, 1, \frac{3}{2}, 0\right), \left(\frac{1}{2}, 2, \frac{1}{2}, 0\right), \\ \left(\frac{3}{2}, 2, \frac{1}{2}, 0\right), \left(\frac{3}{2}, 2, \frac{3}{2}, 0\right).$$

$$YZ: \left(0, \frac{1}{2}, \frac{1}{2}, 0\right), \left(0, \frac{1}{2}, \frac{3}{2}, 0\right), \left(0, \frac{3}{2}, \frac{1}{2}, 0\right), \left(1, \frac{1}{2}, \frac{3}{2}, 1\right), \left(1, \frac{3}{2}, \frac{3}{2}, 1\right), \left(2, \frac{1}{2}, \frac{1}{2}, 0\right), \\ \left(2, \frac{3}{2}, \frac{1}{2}, 0\right), \left(2, \frac{3}{2}, \frac{3}{2}, 0\right).$$

$$YW: \left(1, \frac{1}{2}, 1, \frac{1}{2}\right), \left(1, \frac{1}{2}, 2, \frac{1}{2}\right), \left(1, \frac{3}{2}, 1, \frac{1}{2}\right), \left(1, \frac{3}{2}, 2, \frac{1}{2}\right).$$

$$ZW: \left(1, 0, \frac{3}{2}, \frac{1}{2}\right), \left(1, 2, \frac{3}{2}, \frac{1}{2}\right).$$

These are explicitly the 30 squares in \mathbb{R}^4 corresponding to the disk at the top right of figure 5 (after identifying diametrically opposite points).

Remark 3.3. If S is a gridded surface contained in the scaffolding \mathcal{C}^2 of the canonical cubulation \mathcal{C} of \mathbb{R}^3 , then S is a gridded surface contained in the scaffolding \mathcal{C}^2 of the canonical cubulation \mathcal{C} of \mathbb{R}^4 , since \mathbb{R}^3 is canonically isomorphic to the hyperplane $\mathcal{P} = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$ and \mathcal{P} has a canonical cubulation $\mathcal{C}_{\mathcal{P}}$ given by the restriction of the cubulation \mathcal{C} of \mathbb{R}^4 to it; i.e., $\mathcal{C}_{\mathcal{P}}$ is the decomposition into cubes which are the images of the unit cube $I^3 = \{(x_1, x_2, x_3, 0) \mid 0 \leq x_i \leq 1\}$ by translations by vectors with integer coefficients whose last coordinate is zero.

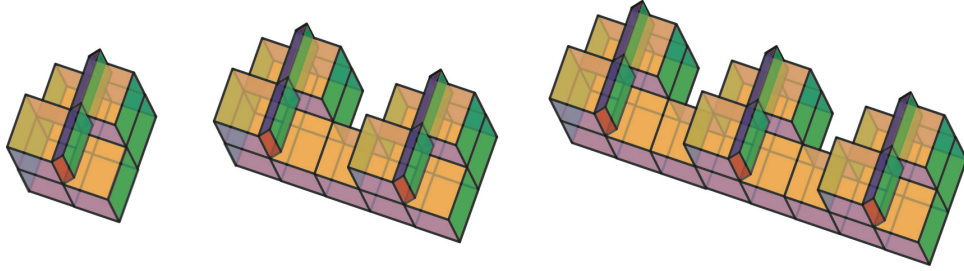


Figure 6: Closed gridded non-orientable surfaces in \mathbb{R}^4 (the first figure is the projection of the crosscap in \mathbb{R}^3 following the description of figure 5).

Let C_1 and C_2 be two gridded surfaces. In a natural way, we define the *gridded connected sum* $\#_S$ of C_1 and C_2 , denoted by $C_1 \#_S C_2$, as follows: We choose embeddings $i_j : \mathbb{D}^2 \rightarrow C_j$ ($j = 1, 2$), such that $i_j(\mathbb{D}^2)$ is a unit square F_i into C_i , $i = 1, 2$. We can assume, up to applying rigid movements that F_1 and F_2 are faces of some 3-cube Q which it does not intersect neither C_1 or C_2 . Thus we obtain $C_1 \#_S C_2$ from the disjoint sum $(C_1 \setminus \text{Int}(Q_1)) \sqcup (C_2 \setminus \text{Int}(Q_2))$ joining F_1 with F_2 via the four remaining faces of Q (see Figure 7). Observe that $C_1 \#_S C_2$ is homeomorphic to the usual connected sum $C_1 \# C_2$.

Lemma 3.4. *If S_1 and S_2 are surfaces homeomorphic to gridded surfaces C_1 and C_2 , respectively; then the connected sum $S_1 \# S_2$ is homeomorphic to the gridded connected sum $C_1 \#_S C_2$.*

Proof. Consider the gridded surfaces C_1 and C_2 . Then $C_1 \#_S C_2$ is homeomorphic to $C_1 \# C_2$. Therefore $C_1 \#_S C_2$ is homeomorphic to $S_1 \# S_2$. \square

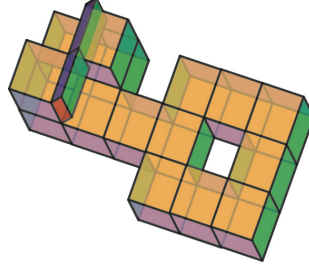


Figure 7: *The gridded connected sum of gridded surfaces is gridded.*

Summarizing, from the above Lemmas and using the classification theorem for closed surfaces, we have the following.

Theorem 3.5. *Any closed surface S is homeomorphic to a gridded surface C such that if S is orientable then C is contained in the scaffolding \mathcal{C}^2 of the canonical cubulation \mathcal{C} of \mathbb{R}^3 , and if S is non orientable then C is contained in the scaffolding \mathcal{C}^2 of the canonical cubulation \mathcal{C} of \mathbb{R}^4 .*

3.2 Surfaces with a finite number of ends in $\{4, 3, 3, 4\}$ in \mathbb{R}^4

We are ready to prove the following theorem.

Theorem 3.6. *Any connected surface with a finite number of ends is homeomorphic to a gridded surface in \mathbb{R}^4 . Moreover any connected orientable surface with a finite number of ends is homeomorphic to a gridded surface in \mathbb{R}^3 .*

Proof. The proof is constructive. We start by constructing a connected orientable gridded surface S with a finite number of ends in \mathbb{R}^3 . The surface S can be of finite or infinite genus. If S has finite genus, say genus g , then all its ends are planar. Let S' be the gridded compact surface of genus g and let E_i ($i = 1, \dots, n$) be all its planar ends, the cylinders; thus $E_i = C_i \times \mathbb{R}$, where $C_i = \partial I^2$ is the boundary of the square. In this case, we can subdivide each square of S if necessary, in such a way that we can choose n squares of S' , say F_1, F_2, \dots, F_n , such that $F_i \cap F_j = \emptyset$ for $i \neq j$. Then, by the classification theorem for noncompact surfaces, S is homeomorphic to the gridded surface $S' \#_S E_1 \#_S \dots \#_S E_n$; where the gridded connected sum of S' with E_i is realized removing the corresponding squares F_i and C_i from S' and E_i , respectively.

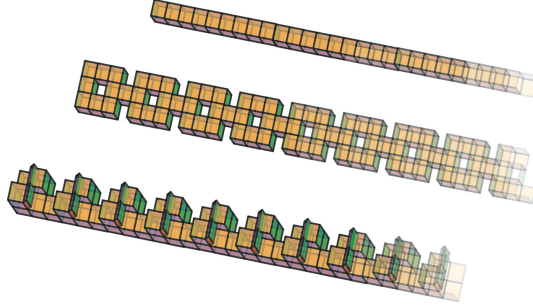


Figure 8: *Types of ends of a surface with finite ends. Cylinder, Job's ladder and an infinite chain of crosscaps*

Suppose now that S has infinite genus. Then S must have non-planar, orientable ends, say E_1, \dots, E_m , $m \geq 1$ (see Figure 8). It can also have planar ends P_{m+1}, \dots, P_n . Consider the gridded orientable infinite genus surface S' obtained as the gridded connected sum of an infinite number of gridded tori (see Figure 8). Notice that by construction, the side at the left first torus of Job's ladder S' consists of tree squares and we call W the corresponding middle square.

We can assume that E_2, \dots, E_m are copies of Jobs's ladder S' . Consider the corresponding middle square $W_i \subset E_i$ (see Figure 8). Let $P_i = C_i \times \mathbb{R}$, where $C_i = \partial I^2$. As in the previous case, we can choose $n - 1$ squares of S' , F_2, F_3, \dots, F_n , such that if we take the 3-ball $B_i(5)$ of radius 5 centered at the barycenter of F_i , then $B_i(5) \cap F_i = \emptyset$ for $i \neq j$. Again by the classification theorem for noncompact surfaces, S is homeomorphic to the gridded surface $S' \#_S E_1 \#_S \dots \#_S E_m \#_S P_{m+1} \#_S \dots \#_S P_n$ where the gridded connected sum of S' with E_i (P_i) is realized removing the corresponding squares F_i and W_i (C_i).

The non-orientable case is analogous, but considering the sets of planar, orientable and non-orientable ends in \mathbb{R}^4 . \square

3.3 Gridded surface with ends of the third derived set in $\{4, 3, 4\}$ in \mathbb{R}^3

There exist gridded connected surfaces with an infinite number of ends in \mathbb{R}^3 , as we will see in the next theorem; however, there exist non-compact surfaces that can not be gridded in \mathbb{R}^3 .

Let's recall that the *derived* set X^1 of a metric space X is the set of all limit points of X . By induction (maybe transfinite induction) we define the n^{th} derived set X^n (or X^ω).

Theorem 3.7. *There are gridded connected surfaces with an infinite number of ends on the third derived set of ends in \mathbb{R}^3 .*

Proof. The proof is constructive. Consider the gridded cylinder $S' = F \times \mathbb{R}$, where $F = \partial\{(x, 0, z) : 0 \leq x, z \leq 1\}$ (see Figure 9).

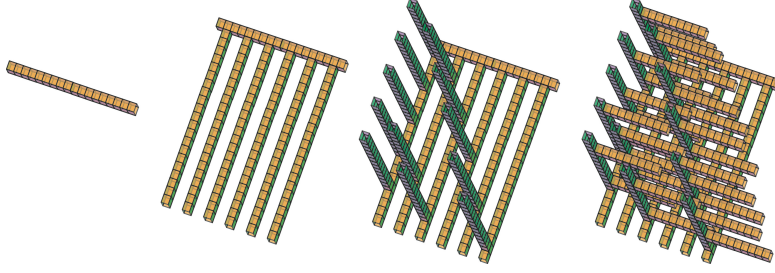


Figure 9: A gridded orientable non compact surfaces.

Let $W_i \subset S'$ be the square $W_i = \{(1, y, z) : i \leq y \leq i + 1, 0 \leq z \leq 1\}$. Let E_i ($i \in \mathbb{Z}$) be planar ends; *i.e.*, $E_i = C_i \times \mathbb{R}$, where $C_i = \partial I^2$. Then the surface $S = S' \#_S E_1 \#_S \dots \#_S E_m \#_S \dots$ is a gridded surface with an infinite number of ends, where the gridded connected sum of C with E_i is realized by removing the corresponding squares $W_{5f(i)}$ and C_i from S' and E_i , respectively; where $f : \mathbb{N} \rightarrow \mathcal{P}$ is a function which sends a natural number n to the corresponding prime number p_n in the set of prime numbers \mathcal{P} (see Figure 9).

Then by this construction, we can describe E_i as $E_i = W_{5f(i)} \times [1, \infty) = \{(x, y, z) : x \geq 1, f(i) \leq y \leq f(i) + 1, 0 \leq z \leq 1\}$. Notice that S' is a limit point on the space of ends; hence S has one end on the first derived set of ends. Now we repeat the same construction at each E_i , *i.e.*, at each E_i we place planar ends $F_{i,j} = C_{i,j} \times \mathbb{R}$ ($i, j \in \mathbb{Z}$) obtaining a gridded surface

$M = S \#_S F_{1_1} \#_S \cdots \#_S F_{1_m} \#_S \cdots$ where the gridded connected sum of S with F_{i_j} is realized by removing the corresponding squares $W_{5f(i), 5f(j)} = \{(r, y, z) : 5f(j) \leq r \leq 5f(j) + 1, 5f(i) \leq y \leq 5f(i) + 1, z = 1\}$ and C_{i_j} from S and F_{i_j} , respectively (see Figure 9). Notice that each E_i is a limit point on the space of ends; in fact, the sequence $\{F_{i_j}\}$ converges to E_i . In other words, M has ends on the second derived set of ends.

Next, we place planar ends $G_{i_{jk}} = C_{i,j,k} \times \mathbb{R}$ ($i, j, k \in \mathbb{Z}$) at each F_{i_j} in the same way to obtain a gridded surface $N = M \#_S G_{1_{1_1}} \#_S \cdots \#_S F_{1_{1_m}} \#_S \cdots$; *i.e.*, where the gridded connected sum of M with $G_{i_{jk}}$ is realized removing the corresponding squares $W_{5f(i), 5f(j), 5f(k)} = \{(x, y, z) : 5f(j) \leq x \leq 5f(j) + 1, 5f(i) \leq y \leq 5f(i) + 1, 5f(k) \leq z \leq 5f(k) + 1\}$ and $C_{i,j,k}$ from M and $G_{i_{jk}}$, respectively (see Figure 9). Observe that each F_{i_j} is a limit point; in fact, the sequence $\{F_{i_{jk}}\}$ converges to F_{i_j} .

Therefore, N has ends on the third derived set of ends. \square

Remark 3.8. The above construction works replacing planar ends by non-planar, orientable ends. It also works for non-orientable surfaces in \mathbb{R}^4 .

3.4 On the impossibility of gridded general surfaces in $\{4, 3^{n-2}, 4\}$ in \mathbb{R}^n

As we have mentioned before, not all the surfaces can be gridded in \mathbb{R}^n for any n .

Theorem 3.9. *There are not gridded connected surfaces in \mathbb{R}^n with a set of ends homeomorphic to the Cantor set.*

Proof. Consider the hypercube $Q = [-m, m]^n$ of length m in \mathbb{R}^n . Then Q contains $(2m)^n$ hypercubes of the canonical cubulation \mathcal{C} of \mathbb{R}^n , hence Q contains $(4m)^n$ squares. In other words, the growth of the number of squares in hypercubes in \mathbb{R}^n is polynomial $((4m)^n)$. On the other hand, the growth of the number of squares required by the tree of life to be gridded is exponential. Therefore, the tree of life is not a gridded surface in $\{4, 3^{n-2}, 4\}$ in \mathbb{R}^n . \square

4 Orientable surfaces in $\{4, 3, 5\}$ in \mathbb{H}^3

All orientable surfaces can be constructed as gridded surfaces on the hyperbolic cubic honeycomb $\{4, 3, 5\}$ of the hyperbolic space \mathbb{H}^3 . We proceed as

in the Euclidean case where we proof that the orientable closed surfaces are gridded in \mathbb{R}^3 by means gridded the torus and the connected sum of two gridded surfaces.

Lemma 4.1. *The torus is a gridded surface in $\{4, 3, 5\}$ in \mathbb{H}^3 . Moreover, all closed orientable surfaces are gridded surfaces in $\{4, 3, 5\}$ in \mathbb{H}^3 .*

Proof. Let the torus be the gridded surface obtained as the boundary of twelve consecutive cubes in $\{4, 3, 5\}$ whose union looks like as O. There are a *central removed cube* and there are 4 cubes around each of its four hyperparallel edges (see Figure 10). This gridded torus in hyperbolic space is the one with the minimum number of squares in the scaffolding of $\{4, 3, 5\}$. It has 44 squares.



Figure 10: *Torus and closed orientable surfaces in $\{4, 3, 5\}$ in \mathbb{H}^3 .*

There is a completely analogous hyperbolic concept of connected sum for gridded surfaces as in the previous Euclidean section. Let S_1 and S_2 gridded surfaces in \mathbb{H}^3 and $D_1 \subset S_1$ and $D_2 \subset S_2$ two squares such that each one of the corresponding support hyperbolic plane \hat{D}_i $i = 1, 2$, divides \mathbb{H}^3 in two half-spaces in such a way that S_i is contained in only one half-space. Then we can construct the connected sum $S_1 \# S_2$ as a gridded surface. A closed orientable surface can be gridded in this hyperbolic context as a connected sum of gridded tori. \square

A *hyperbolic pair of pants* is a closed pair of pants with a hyperbolic metric such that each of its three boundary circles are geodesics. The isometry class of such pair of pants is determined by the triple of lengths (l_1, l_2, l_3) of the boundaries.

Lemma 4.2. *The pair of pants is a gridded surface in $\{4, 3, 5\}$ in \mathbb{H}^3 .*

Proof. Let the pair of pants be the gridded surface obtained as the boundary of four cubes in $\{4, 3, 5\}$ whose union looks like as T minus three squares. There are a *central cube* C and three neighborhood cubes C_i of it such

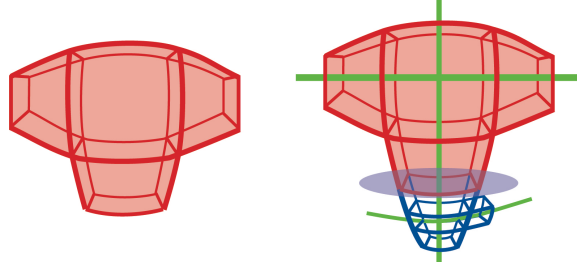


Figure 11: A gridded pair of pants in $\{4, 3, 5\}$ in \mathbb{H}^3 .

that these four cubes do not share a vertex; *i.e.* $C \cap \cup_{i=1}^3 C_i = \emptyset$. Then $C \# C_1 \# C_2 \# C_3$ is our pair of pants (see Figure 11). \square

Lemma 4.3. *The tree of life is a gridded surface in $\{4, 3, 5\}$ in \mathbb{H}^3 .*

Proof. The proof is constructive by means pair of pants pasted along their boundaries as an infinite tree.

There are two distinguished geodesics in our model of the pair of pants. Notice that a pair of pants has a rotational symmetry of order 2. The *axis of symmetry* is a geodesic which pass through the barycenters of the central cube and the second neighborhood cube *i.e.* the vertical bar in the T. The *second axis* is the geodesic perpendicular to the axis of symmetry which passes through the barycenters of the central cube and the first and third neighborhood cubes *i.e.* the horizontal bar in the T. The axis of symmetry is ultraparallel to the two squares which were removed from the first and third neighborhood cubes and the second axis is ultraparallel to the square which was removed from the second neighborhood cube.

Then we can construct inductively the tree of life. We start by a pair of pants P and glue it both a square in the boundary of the second cube and two pairs of pants P_i at each boundary of the first and third cubes of P with the second neighborhood cubes of the corresponding P_i ; in such away that all barycenters of the cubes are lie in one hyperbolic plane (see Figure 11). The second axis of the first pair of pants is the axis of symmetry of the two pair of pants which have been glued to its first and third neighborhood cubes. The second axis of the new two cubes is ultraparallel to the axis of symmetry of the original cube and these geodesics are ultraparallel to the hyperbolic plane defined by the square of gluing of each two pair of pants.

This plane divides the tree of life in two connected components.

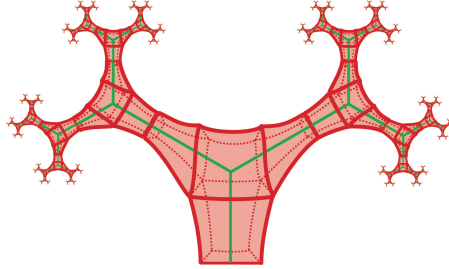


Figure 12: *Gridded Tree of life in $\{4, 3, 5\}$ in \mathbb{H}^3 .*

We glue new pairs of pants by the second neighborhood cube in the boundaries of the surface such that all barycenters of the cubes are in the hyperbolic plane. The step of induction is that the second axis of the pair of pants in the surface is the axis of symmetry of the two pair of pants which have been glued to their first and third neighborhood cubes. The second axis of the new two cubes is ultraparallel to the axis of symmetry of the original cube and these geodesics are ultraparallel to the hyperbolic plane defined by the square of gluing of each two pair of pants. This plane divides the tree of life in two connected components. By induction, we have constructed the tree of life. \square

We are ready to prove the following theorem.

Theorem 4.4. *Any connected orientable surface is homeomorphic to a gridded surface in $\{4, 3, 5\}$ in \mathbb{H}^3 .*

Proof. Any connected orientable surface can be constructed from the pruned tree of life replacing some pair of pants by “handles”. We can consider the gridded torus minus three nonconsecutive equatorial squares. Notice that when we exchange a pair of pants by these gridded tori minus three nonconsecutive equatorial squares, the property of the gridded connected sum is preserved. The hyperbolic planes which pass through the boundaries of the modified pair of pants are ultraparallels, then the construction of a tree of life with handles is analogous to the planar tree of life.

5 Surfaces in $\{4, 3, 3, 5\}$ in \mathbb{H}^4

One great difference between the Euclidean and the hyperbolic gridded cases is that the gridded Euclidean spaces are nested and the gridded hyperbolic spaces are not (as gridded spaces). For the Euclidean case we prove that the orientable closed surfaces are gridded in \mathbb{R}^3 by means gridded the torus and the connected sum. We proved that the projective plane is gridded in \mathbb{R}^4 and all closed surfaces are gridded in \mathbb{R}^4 .

The gridded hyperbolic 3-space $\{4, 3, 5\}$ is not contained in the gridded hyperbolic 4-space $\{4, 3, 3, 5\}$. However, it is not a great problem to grid in \mathbb{H}^4 all the gridded surfaces in \mathbb{H}^3 . We need to grid the torus, the pair of pants and the connected sum of gridded surfaces in order to obtain all orientable surfaces. For nonorientable surfaces we need only to prove that the projective plane is gridded in \mathbb{H}^4 and applying the Richards Theorem then we will obtain all surfaces gridded in \mathbb{H}^4 .

Lemma 5.1. *The torus is a gridded surface in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 . Moreover, all closed orientable surfaces are gridded surfaces in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

Proof. There are 8 cubes in the hypercube forming two linked tori in \mathbb{S}^3 (see Figure 13). We take one of these torus. Let the torus be the gridded surface obtained as the boundary of four consecutive cubes in a hypercube $\{4, 3, 3\}$.

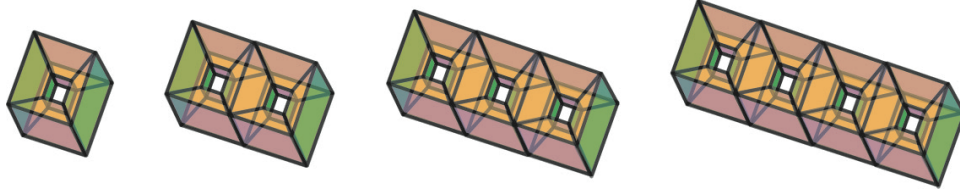


Figure 13: Gridded torus and closed orientable surfaces in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .

As above, we have a 4-dimensional analogous hyperbolic concept of connected sum for gridded surfaces. Let S_1 and S_2 be two hyperbolic gridded surfaces in \mathbb{H}^4 and $D_i \subset S_i$, $i = 1, 2$ be squares such that the corresponding support hyperbolic plane \hat{D}_i lies in a 3-dimensional geodesic hyperbolic subspace which divides \mathbb{H}^4 in two half-spaces such that S_i is contained in only one half-space. Then we can construct the connected sum $S_1 \# S_2$ as a gridded surface. A closed orientable surface can be gridded in this hyperbolic context as a connected sum of gridded tori. \square

Lemma 5.2. *The pair of pants is a gridded surface in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

Proof. Consider three 3-faces F_1 , F_2 and F_3 of three consecutive hypercubes C_1 , C_2 and C_3 such that $F_i \subset C_i$ ($i = 1, 2, 3$) and $F_1 \cap F_2$ and $F_2 \cap F_3$ are two disjoint squares. The barycenters of C_1 , C_2 and C_3 are collinear. Consider the connected sum $F_1 \# F_2 \# F_3$. Notice that $F_1 \# F_2 \# F_3$ is homeomorphic to a \mathbb{S}^2 . Then the pair of pants is obtained from $F_1 \# F_2 \# F_3$ by removing one boundary square S_i from F_i , in such a way that S_1 is parallel to $F_1 \cap F_2$ and S_3 is parallel to $F_2 \cap F_3$ (see Figure 14). \square

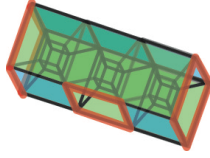


Figure 14: *The gridded pair of pants in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

Lemma 5.3. *The tree of life is a gridded surface in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

Proof. The proof is constructive using pair of pants pasted along their boundaries as an infinite tree.

There are two important geodesics in our model of a pair of pants. A pair of pants has a rotational symmetry of order 2. The *axis of symmetry* is a geodesic which passes through the barycenter of F_2 and the barycenter of the square S_2 . The *second axis* is the geodesic perpendicular to the axis of symmetry which passes through the barycenters of S_1 and S_3 . The axis of symmetry is ultraparallel to the two squares which were removed from F_1 and F_3 and the second axis is ultraparallel to the square which was removed from F_2 .

We can construct inductively the tree of life in analogous way as \mathbb{H}^3 . \square

Lemma 5.4. *The projective plane is a gridded surface in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

Proof. We construct a gridded version of the crosscap in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 . See the Figures 5 and 15. In the left we show the projection of the crosscap. In the middle we divide the crosscap in three parts: at the bottom, there is the base which is a pentagonal cubic box minus two squares at the top. In the middle, there is a band and at the top there is a disk which is a neighborhood of one vertex. Only the base is different in the two Figures 5 and 15.

In the right part of the Figure 15 there is a description of the combinatorial square complex of the crosscap as a disk with 34 squares after identifying points in the circle boundary by the antipodal map. At the bottom, there is a Möbius band contained in this crosscap. This is similar to the Euclidean case except one uses 4 more squares in the crosscap (see figure 5).

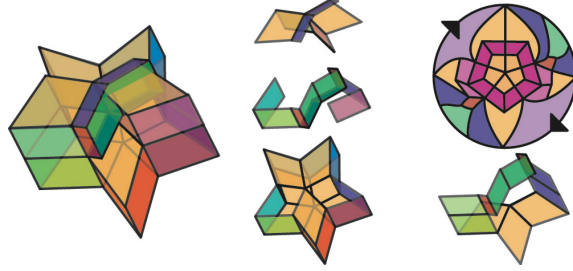


Figure 15: *Gridded projective plane in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

We are ready to prove the following theorem.

Theorem 5.5. *Any connected surface is homeomorphic to a gridded surface in $\{4, 3, 3, 5\}$ in \mathbb{H}^4 .*

Proof. Any connected surface can be constructed from the pruned tree of life modifying some pair of pants by means put “handles” and “projective planes”. Consider the connected sum of a hyperbolic gridded torus in $\{4, 3, 3, 5\}$ and the pair of pants. In fact, from a gridded torus we remove an open square and we paste the boundary of this square onto the boundary of a removed square in the central cube F_2 of the pair of pants (see Figure 16).

As above, we consider the connected sum of a hyperbolic gridded projective plane in $\{4, 3, 3, 5\}$ and the pair of pants. Remove from the gridded projective plane an open square in its base and paste its boundary with the boundary of one square in the central cube F_2 of the pair of pants.

Notice that, if we exchange a pair of pants of a pruned tree of life by these kind of new pair of pants the property of the gridded connected sum is preserved. The hyperbolic spaces which pass by the boundaries of these new pair of pants are ultraparallels and divide \mathbb{H}^4 in two half-spaces where the pair of pants is contained in one component. Then the construction of a tree of life with handles and projective planes is analogous to the construction of an orientable noncompact surface in \mathbb{H}^3 (see Theorem 4.4).

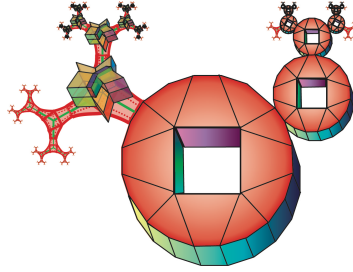


Figure 16: *Tree of life with handles and projective planes. The non-planar non orientable ends go to the top left and the orientable non-planar ends go to the top right.*

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